

SLOPE PROBLEM FOR TRAJECTORIES OF HOLOMORPHIC SEMIGROUPS IN THE UNIT DISC

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ABSTRACT. It has been an open problem for about ten years whether every trajectory of a parabolic one-parameter semigroup in the unit disk tends to the Denjoy – Wolff point with a definite (and common for all trajectories) slope. In this paper, we give the negative answer to this question.

1. INTRODUCTION

A *one-parameter semigroup of holomorphic functions* in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ is a continuous homomorphism $t \mapsto \varphi_t$ between the semigroup of non-negative real numbers endowed with the Euclidean topology and the semigroup $\mathbf{Hol}(\mathbb{D}, \mathbb{D})$ of all holomorphic self-maps of the unit disc endowed with the topology of uniform convergence on compacta.

The study of this kind of one-parameter semigroups has a long history. They have appeared in many different areas of Analysis such as operator theory, geometric function theory, optimization and branching stochastic processes, see, *e.g.*, [13]. Moreover, one-parameter semigroups of holomorphic functions can be regarded as a special but very important case (namely, the autonomous case from the dynamical point of view) of a more general notion playing a fundamental role in Loewner theory, a current and very active branch of research (see, *e.g.*, the recent survey [2]). We refer the reader to the monographs [17] and [10] for the basic definitions and results of the theory of one-parameter semigroups.

In the last decades, starting with the seminal paper by Berkson and Porta [3], the study of the dynamical aspects of the theory has attracted considerable interest. Namely, for any one-parameter semigroup $(\varphi_t) \subset \mathbf{Hol}(\mathbb{D}, \mathbb{D})$, there exists a *unique* analytic function $G : \mathbb{D} \rightarrow \mathbb{C}$ (the so-called vector field or *infinitesimal generator* of the semigroup) such

Date: June 16, 2014.

2000 *Mathematics Subject Classification.* Primary 30C80; Secondary 30D05, 30C35, 34M15.

Key words and phrases. Semigroups of analytic functions, infinitesimal generator, Koenigs function.

[†] Partially supported by the *Ministerio de Economía y Competitividad* and the European Union (FEDER), project MTM2012-37436-C02-01, and by *La Consejería de Educación y Ciencia de la Junta de Andalucía*.

[‡] Partially supported by the FIRB grant Futuro in Ricerca “Geometria Differenziale Complessa e Dinamica Olomorfa” n. RBFR08B2HY.

that, for each $z \in \mathbb{D}$, the positive trajectory associated with the Cauchy problem

$$\begin{cases} \dot{w} = G(w), \\ w(0) = z, \end{cases}$$

is exactly $[0, +\infty) \ni t \mapsto \varphi_t(z)$. This fact gives rise, in a natural way, to a number of dynamical questions concerning the complete (not only the positive) trajectories of the vector field G and thus leading to dynamical questions for the corresponding semigroup (φ_t) . This dynamical approach includes topics like the asymptotic behavior of the trajectories, their ω -limits, the analysis of the boundary fixed points, the multipliers of these points or the slopes of the trajectories arriving to a boundary fixed point. It is worth mentioning that this dynamical point of view (in particular, analysis of slopes) has also been extensively treated for holomorphic discrete iteration in the unit disk (see, *e.g.*, the papers by Poggi-Corradini [14] and Pommerenke [15]).

We recall that a point ξ in the Riemann Sphere \mathbb{C}_∞ is called an ω -*limit point* of a curve $\Gamma : (s_1, s_2) \rightarrow \mathbb{C}$, $-\infty < s_1 < s_2 \leq +\infty$, if there exists a strictly increasing sequence $(t_n) \subset (s_1, s_2)$ convergent to s_2 such that $\Gamma(t_n) \rightarrow \xi$. The set of all ω -limit points of Γ is called the ω -*limit set* and denoted by $\omega(\Gamma)$.

Concerning the asymptotic behavior, the most fundamental result is the continuous version of the Denjoy–Wolff theorem, see, *e.g.*, [17, Proposition 4.1.2], stating basically that for any one-parameter semigroup $(\varphi_t) \subset \text{Hol}(\mathbb{D}, \mathbb{D})$ such that φ_{t_0} has no fixed point in \mathbb{D} for some $t_0 > 0$, there exists a (unique) point $\tau \in \partial\mathbb{D}$, called the *Denjoy–Wolff point* of (φ_t) , such that

$$\lim_{t \rightarrow +\infty} \varphi_t(z) = \tau \quad \text{for every } z \in \mathbb{D}.$$

Moreover, see, *e.g.*, [17, Theorem 2.12], $\lim_{r \rightarrow 1^-} \varphi'_t(r\tau) = \exp(-\lambda t)$ for some $\lambda \geq 0$ and all $t \geq 0$. The semigroup (φ_t) is said to be *hyperbolic* if $\lambda > 0$, and *parabolic* if $\lambda = 0$.

The continuous Denjoy–Wolff theorem opens the door to the study of the slopes of trajectories with respect to the Denjoy–Wolff point. Namely, given $z \in \mathbb{D}$, one defines the *set of slopes* $\text{Slope}^+(\varphi_t(z), \tau)$ of a (hyperbolic or parabolic) semigroup (φ_t) with associated Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ as the ω -limit of the curve

$$[0, +\infty) \ni t \mapsto \text{Arg}(1 - \overline{\tau}\varphi_t(z)) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

where Arg denotes the principal argument. It is well-known that $\text{Slope}^+(\varphi_t(z), \tau)$ is a (non-empty) closed subinterval of $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

If the semigroup (φ_t) is hyperbolic, it is known [4, Theorem 2.8] that, for every $z \in \mathbb{D}$, $\text{Slope}^+(\varphi_t(z), \tau)$ is a single point of $(-\pi/2, \pi/2)$ and

$$\bigcup_{z \in \mathbb{D}} \text{Slope}^+(\varphi_t(z), \tau) = (-\pi/2, \pi/2).$$

In the parabolic case, the situation is quiet different. On the one hand, it is possible to prove that $\text{Slope}^+(\varphi_t(z), \tau)$ does not depend on z [4, Theorem 2.9]. On the other hand,

there is no such a strong result as in the hyperbolic case and, indeed, there has been open for a long time the so-called

“The Parabolic Slope Conjecture”:

There exists $\theta \in [-\pi/2, \pi/2]$ such that $\text{Slope}^+(\varphi_t(z), \tau) = \{\theta\}$ for all $z \in \mathbb{D}$.

This conjecture has been studied and explicitly mentioned in many papers ([4], [7], [8], [9], [11], [12]), stating its validity in a lot of cases. To our best knowledge, the best two relevant results (of somehow different nature) are:

- if (φ_t) is a parabolic one-parameter semigroup with positive hyperbolic step, then $\text{Slope}^+(\varphi_t(z), \tau)$ is a single point and it is equal to $-\pi/2$ or $\pi/2$, see [4] and [5];
- if for some $\varepsilon > 0$ the limit

$$\lim_{z \rightarrow \tau} \frac{G(z)}{(z - \tau)^{1+\varepsilon}}$$

exists finitely and it is different from zero¹, then $\text{Slope}^+(\varphi_t(z), \tau)$ is a single point [7].

In other words, with a minor assumption of regularity, the above conjecture is true. We, however, prove below that the conjecture is false in general.

Theorem 1.1. *There exists a parabolic semigroup (φ_t) in the unit disk with the Denjoy–Wolff point $\tau = 1$ such that*

$$\text{Slope}^+(\varphi_t(z), 1) = [-\pi/2, \pi/2], \text{ for each } z \in \mathbb{D}.$$

In particular, “The Parabolic Slope Conjecture” is false.

Note: After finishing the redaction of this paper, Prof. D. Betsakos kindly communicated to the authors that he also had just obtained a counterexample to the conjecture using a different approach, based on certain estimates for the harmonic measure.

2. PROOF OF THE MAIN RESULT

Proof. We start by introducing some notations. Given four real numbers u_1, u_2, v, w with $u_1 < u_2$ and $v, w > 0$, we define the associated rectangle

$$R(u_1, u_2, v, w) := \{z \in \mathbb{C} : u_1 < \text{Re } z \leq u_2, -w < \text{Im } z < v\}.$$

This kind of rectangles will be the bricks to generate a wide family of domains which will be indexed by the set $I \times J \times J$, where:

- I is the set of all strictly increasing sequences of positive real numbers converging to $+\infty$.
- J is the set of all increasing (not necessarily strictly) sequences of real numbers with values in $[1, +\infty)$.

¹Notice that for any parabolic one-parameter semigroup, $\angle \lim_{z \rightarrow \tau} \frac{G(z)}{z - \tau} = 0$.

Then, given $\lambda = (\alpha, \beta, \gamma) \in I \times J \times J$ with $\alpha = (u_n)$, $\beta = (v_n)$ and $\gamma = (w_n)$, we set

$$\Omega_\lambda := \mathbb{R}(-1, u_1, 1, 1) \cup \left(\bigcup_{j=1}^{\infty} \mathbb{R}(u_j, u_{j+1}, v_j, w_j) \right).$$

We note that Ω_λ is a simply connected domain of the complex plane and $[0, +\infty) \subset \Omega_\lambda$. This allows us to consider the canonical Riemann map $h_\lambda = h_{\Omega_\lambda}$ from \mathbb{D} onto Ω_λ normalized, as usual, by

$$h_\lambda(0) = 0, \quad h'_\lambda(0) > 0.$$

Moreover, by [16, Section 9.7, Lemma 2], there exists

$$\lim_{t \rightarrow +\infty} h_\lambda^{-1}(t) =: b_\lambda \in \partial\mathbb{D}.$$

In particular, the map $\mathbb{D} \ni z \mapsto g_{\Omega_\lambda}(z) = g_\lambda(z) := h_\lambda(b_\lambda z)$ is another conformal map of \mathbb{D} onto Ω_λ fixing the origin. The key difference with h_λ is that the normalization now comes from the fact

$$\lim_{t \rightarrow +\infty} g_\lambda^{-1}(t) = 1.$$

Since

$$\Omega_\lambda + t \subset \Omega_\lambda \quad \text{for every } t \geq 0,$$

we may define the mappings

$$\varphi_t^{\Omega_\lambda}(z) := g_\lambda^{-1}(g_\lambda(z) + t), \quad t \geq 0, \quad z \in \mathbb{D}.$$

Bearing in mind the normalization of g_λ , we deduce that $(\varphi_t^{\Omega_\lambda})$ is a one-parameter semi-group of holomorphic functions in \mathbb{D} having the Denjoy–Wolf point at 1 and with g_λ as its Koenigs function.

A fundamental property of this family of domains $\mathcal{D} := (\Omega_\lambda)_{\lambda \in I \times J \times J}$ is compiled in the following claim, whose proof will be given at the end.

Claim: *Fix $n \in \mathbb{N}$ and the following three vectors:*

$$\begin{aligned} &(u_1, \dots, u_{n+1}) \text{ with } 0 < u_1 < u_2 < \dots < u_{n+1}, \\ &(v_1, \dots, v_n) \text{ with } 1 \leq v_1 \leq v_2 \leq \dots \leq v_n, \\ &(w_1, \dots, w_n) \text{ with } 1 \leq w_1 \leq w_2 \leq \dots \leq w_n. \end{aligned}$$

Then, for any $\varepsilon \in (0, 1)$, there exists $M_1 = M_1(\varepsilon) > 0$ (resp. $M_2 = M_2(\varepsilon) > 0$) such that for any $M \geq M_1$ (resp. $M \geq M_2$) and any $\Omega \in \mathcal{D}$ satisfying

$$(2.1) \quad \{z \in \Omega : \operatorname{Re} z \leq u_{n+1} + M\} = \mathbb{R}(-1, u_1, 1, 1) \cup \left(\bigcup_{j=1}^{n+1} \mathbb{R}_j \right),$$

where

$$\begin{aligned} \mathbb{R}_j &:= \mathbb{R}(u_j, u_{j+1}, v_j, w_j) \text{ for } j = 1, \dots, n, \text{ and} \\ \mathbb{R}_{n+1} &:= \mathbb{R}(u_{n+1}, u_{n+1} + M, v_n + M, w_n) \text{ (resp. } \mathbb{R}_{n+1} := \mathbb{R}(u_{n+1}, u_{n+1} + M, v_n, w_n + M)), \end{aligned}$$

we have

$$\operatorname{Arg}(1 - g_\Omega^{-1}(\xi)) \geq \frac{\pi}{2}(1 - \varepsilon) \quad (\text{resp. } \operatorname{Arg}(1 - g_\Omega^{-1}(\xi)) \leq -\frac{\pi}{2}(1 - \varepsilon))$$

for some $\xi \in (u_{n+1}, u_{n+1} + M)$.

Assuming this claim, we can generate recurrently three sequences

$$\alpha = (u_n) \in I, \quad \beta = (v_n) \in J, \quad \gamma = (w_n) \in J,$$

$\lambda := (\alpha, \beta, \gamma)$, the corresponding domain $\Omega := \Omega_\lambda \in \mathcal{D}$ and an additional sequence of real numbers (ξ_n) in such a way that

- (A) $\lim_n v_n = \lim_n w_n = +\infty$,
- (B) $u_n < \xi_n < u_{n+1}$, for every $n \in \mathbb{N}$,
- (C) $\operatorname{Arg}(1 - g_\Omega^{-1}(\xi_{2n})) \leq -\frac{\pi}{2}(1 - \frac{1}{2n})$ for every $n \in \mathbb{N}$,
- (D) $\operatorname{Arg}(1 - g_\Omega^{-1}(\xi_{2n+1})) \geq \frac{\pi}{2}(1 - \frac{1}{2n})$ for every $n \in \mathbb{N}$.

In particular, the ω -limit of the curve

$$[0, +\infty) \ni t \mapsto \operatorname{Arg}(1 - \varphi_t^\Omega(0)) = \operatorname{Arg}(1 - g_\Omega^{-1}(t))$$

contains the numbers $\pi/2$ and $-\pi/2$. Therefore, $\operatorname{Slope}^+(\varphi_t^\Omega(0), 1) = [-\pi/2, \pi/2]$. Finally, because of (A) and appealing to [4, Theorem 2.9 (1)], we obtain that

$$\operatorname{Slope}^+(\varphi_t^\Omega(z), 1) = [-\pi/2, \pi/2] \quad \text{for every } z \in \mathbb{D}.$$

The fact that the semigroup (φ_t^Ω) is parabolic follows from (A) by [4, Corollary 2.2].

Proof of the Claim. We only deal with one of the two possibilities (the other is completely similar) and proceed by reductio ad absurdum. So we suppose on the contrary to our claim that there exist $\varepsilon > 0$, a sequence of positive real numbers (M_k) with $\lim_k M_k = +\infty$ and a family of domains (Ω_k) in \mathcal{D} such that, for every $k \in \mathbb{N}$,

- (1) equality (2.1) holds for $\Omega := \Omega_k$ and $R_{n+1} := R(u_n, u_{n+1} + M_k, v_n + M_k, w_n)$,
- (2) $\operatorname{Arg}(1 - g_{\Omega_k}^{-1}(\xi)) < \frac{\pi}{2}(1 - \varepsilon)$ for all $\xi \in (u_{n+1}, u_{n+1} + M_k)$.

It is easy to see that the sequence (Ω_k) converges to its kernel (w.r.t. the origin) in the sense of Carathéodory,

$$\Omega_k \rightarrow \widehat{\Omega} := \operatorname{Ker}_0\{\Omega_k\} = \{z \in \mathbb{C} : \operatorname{Re} z > u_{n+1}, \operatorname{Im} z > -w_n\} \cup \left(\bigcup_{j=1}^n R_j \right).$$

Let h_k and h stand for the canonical Riemann maps of \mathbb{D} onto Ω_k and Ω , respectively, normalized, as usual, by $h_k(0) = h(0) = 0$, $h'_k(0) > 0$, $h'(0) > 0$. Then by the Carathéodory kernel convergence theorem, see, e.g., [6, Chapter 15, Corollary 4.11 and Theorem 4.7],

- (i) $h_k \rightarrow h$ uniformly on compacta as $k \rightarrow +\infty$,
- (ii) for each compact set $K \subset \widehat{\Omega}$, all but a finite number of h_k^{-1} 's are defined on K and $h_k^{-1} \rightarrow h^{-1}$ uniformly on K as $k \rightarrow +\infty$.

Using again [16, Section 9.7, Lemma 2], we obtain

$$b_k = \lim_{t \rightarrow +\infty} h_k^{-1}(t) \in \partial\mathbb{D}, \quad k \in \mathbb{N}, \quad b = \lim_{t \rightarrow +\infty} h^{-1}(t) \in \partial\mathbb{D}.$$

Moreover, $b = \lim_k b_k$, essentially due to Julia's Lemma. Indeed, since $h(\mathbb{D}) + t \subset h(\mathbb{D})$ and $h(0) = 0$, we can consider the associated one-parameter semigroup of the disk $\phi_t := h^{-1} \circ (h + t)$. Using (i) and (ii), we find that ϕ_t is the pointwise limit of ϕ_t^k , where (ϕ_t^k) is the one-parameter semigroup associated with the couple (Ω_k, h_k) , *i.e.*, $\phi_t^k := h_k^{-1} \circ (h_k + t)$. Now passing to the limit in Julia's inequality [1, (1.2.9) on p. 49], we see that if μ is an accumulation point of (b_k) , then

$$\frac{|\phi_t(z) - \mu|^2}{1 - |\phi_t(z)|^2} \leq \frac{|z - \mu|^2}{1 - |z|^2}, \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0,$$

from which it follows that $\mu = b$, because $\phi_t(z) \rightarrow b$ as $t \rightarrow +\infty$ for all $z \in \mathbb{D}$ according to the continuous Denjoy–Wolff theorem.

Therefore, defining $g(z) := h(bz)$, $z \in \mathbb{D}$, and denoting $g_k := g_{\Omega_k}$, we also have that $g_k^{-1} \rightarrow g^{-1}$, as $k \rightarrow +\infty$, uniformly on compact subsets of $\widehat{\Omega}$ in the same sense as in (ii).

Clearly g is the Koenigs function of the one-parameter semigroup (ψ_t) , given by $\psi_t(z) := \bar{b}\phi_t(bz)$ for all $z \in \mathbb{D}$ and $t \geq 0$. By [4, Corollary 2.2 and Theorem 2.9 (2)], we deduce that the limit

$$(2.2) \quad \theta_0 := \lim_{t \rightarrow +\infty} \text{Arg}(1 - g^{-1}(t))$$

exists and must value $\pi/2$ or $-\pi/2$. We will prove now that both cases are impossible.

Assume first that $\theta_0 = \pi/2$. In this case there would exist $T > u_{n+1}$ satisfying $\text{Arg}(1 - g^{-1}(T)) > \frac{\pi}{2}(1 - \frac{\varepsilon}{2})$. Since $g_k^{-1} \rightarrow g^{-1}$ and $M_k \rightarrow +\infty$ as $k \rightarrow +\infty$, we could also find $N \in \mathbb{N}$ such that $u_{n+1} + M_N > T$ and $\text{Arg}(1 - g_N^{-1}(T)) > \frac{\pi}{2}(1 - \varepsilon)$, which would contradict (2). This shows that $\theta_0 \neq \pi/2$.

Suppose now that the limit (2.2) equals $-\pi/2$. Set $\omega := \frac{1+i}{\sqrt{2}}$ and consider the unique conformal map f of \mathbb{D} onto $\Omega_0 := \{z \in \mathbb{C} : \text{Re } z > u_{n+1}, \text{Im } z > -w_n\}$ such that $f(0) = u_{n+1} - iw_n + \omega$ and $\lim_{u_{n+1} < t \rightarrow +\infty} f^{-1}(t) = 1$. In fact, f is given by the formula

$$f(z) = \frac{1+i}{\sqrt{2}} \sqrt{\frac{1+z}{1-z}} + u_{n+1} - iw_n, \quad z \in \mathbb{D},$$

where the square root is to be understood as its principle branch. In particular, it is easy to check that

$$(2.3) \quad \lim_{u_{n+1} < t \rightarrow +\infty} \text{Arg}(1 - f^{-1}(t)) = \frac{\pi}{2}.$$

By construction, $f(\mathbb{D}) \subset \widehat{\Omega}$. Moreover, $\widehat{\Omega}$ and $f(\mathbb{D})$ are Jordan domains, so g and f both extend homeomorphically to the closed unit disk. Therefore, $\phi := g^{-1} \circ f$ is a holomorphic self-map of \mathbb{D} that also extends homeomorphically to $\overline{\mathbb{D}}$. Recall that $\lim_{t \rightarrow +\infty} g^{-1}(t) = 1$. Hence $\phi(1) = 1$. Moreover, bearing in mind the shape of the (Jordan) domains $f(\mathbb{D})$ and $\widehat{\Omega}$,

we can apply the Schwarz's reflection principle to extend ϕ holomorphically through an open arc of $\partial\mathbb{D}$ containing the point 1. Note that the extended map is univalent. It follows that

$$(2.4) \quad \frac{1 - \phi(z)}{1 - z} \rightarrow \phi'(1) > 0 \text{ as } \mathbb{D} \ni z \rightarrow 1 \quad (\text{the unrestricted limit in } \mathbb{D}).$$

Finally, consider the following equality, for $t > 0$ large enough,

$$\frac{1 - g^{-1}(t)}{|1 - g^{-1}(t)|} = \frac{1 - \phi(f^{-1}(t))}{1 - f^{-1}(t)} \frac{1 - f^{-1}(t)}{|1 - f^{-1}(t)|} \frac{|1 - f^{-1}(t)|}{|1 - \phi(f^{-1}(t))|}$$

On the one hand, since the limit (2.2) is $-\pi/2$, the left-hand side must tend to $-i$ as $t \rightarrow +\infty$. On the other hand, by (2.3) and (2.4), the right-hand side tends to i as $t \rightarrow +\infty$. This contradiction completes the proof of the Claim. \square

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